Constrained Accelerations for Controlled Geometric Reduction: Sagittal-Plane Decoupling for Bipedal Locomotion

Robert D. Gregg, Ludovic Righetti, Jonas Buchli and Stefan Schaal

Abstract—Energy-shaping control methods have produced strong theoretical results for asymptotically stable 3D bipedal dynamic walking in the literature. In particular, geometric controlled reduction exploits robot symmetries to control momentum conservation laws that decouple the sagittal-plane dynamics, which are easier to stabilize. However, the associated control laws require high-dimensional matrix inverses multiplied with complicated energy-shaping terms, often making these control theories difficult to apply to highly-redundant humanoid robots. This paper presents a first step towards the application of energy-shaping methods on real robots by casting controlled reduction into a framework of constrained accelerations for inverse dynamics control. By representing momentum conservation laws as constraints in acceleration space, we construct a general expression for desired joint accelerations that render the constraint surface invariant. By appropriately choosing an orthogonal projection, we show that the unconstrained (reduced) dynamics are decoupled from the constrained dynamics. Any acceleration-based controller can then be used to stabilize this planar subsystem, including passivity-based methods. The resulting control law is surprisingly simple and represents a practical way to employ control theoretic stability results in robotic platforms. Simulated walking of a 3D compass-gait biped show correspondence between the new and original controllers, and simulated motions of a 16-DOF humanoid demonstrate the applicability of this method.

I. INTRODUCTION

Dynamic walking robots have been successful at exploiting natural dynamics to produce energy-efficient gaits [1], [9], [15]. However, these robots lack the same functionality and utility as inefficient humanoids like ASIMO and HRP-2 for a variety of reasons, e.g., some depend on downhill slopes for passive walking, are constrained to the sagittal plane-of-motion, lack directional control authority, and/or lack redundant joints for manipulation.

Many of these limitations are attributable to challenges in control law design for asymptotically stable walking with high-dimensional bipedal robots. Energy-shaping and passivity-based control approaches have proven useful in theoretical studies by exploiting zero-cost passive limit cycles that naturally appear down shallow slopes in certain mechanical systems [9]. These gaits can be mapped to arbitrary slopes by shaping the potential energy for a slope-invariant controlled symmetry [12], or mapped into faster time scales by also shaping the kinetic energy [13]. Although passive limit cycles are very sensitive to perturbations, passivity-based energy regulation can provide expanded basins of attraction for robustness on uneven terrain [13]. However, these methods require the existence of stable passive limit cycles, which is usually not the case for simple bipeds in 3D space, let alone highly-redundant humanoids.

Recent work on controlled geometric reduction has shown promise in addressing some of these problems for bipedal dynamic walking. This energy-shaping approach exploits robot symmetries to impose controlled momentum conservation laws that decouple sagittal-plane dynamics [2], [5], which are easier to stabilize using passivity-based or trajectory-tracking methods. Asymptotic stability of the sagittal-plane dynamics then implies asymptotic stability of the non-sagittal dynamics by the conservation laws. Numerical simulations have shown that this produces asymptotically stable straight-ahead and turning gaits in 3D space [2], [4], enabling stable path planning via dynamic walking primitives [3].

However, controlled reduction is designed in stages, providing dynamical decoupling one DOF at a time, and requires a sophisticated control law to insert complicated shaping terms for each reduction stage. Moreover, energy-shaping control laws in general require a full inverse of the desired inertia matrix. While attractive from a theoretical standpoint, this complexity does not scale well to high-dimensional robotic systems (in terms of computation time and modeling error). This may explain why energy-shaping methods have not been applied in practice to redundant humanoid robots.

More standard control methods such as inverse dynamics or operational space control have proven to be efficient and practical for compliant control of redundant robots [8], [11]. Such methods were, for example, key to the control of the Little Dog robot over rough terrains [7], [10]. The redundant inverse kinematics scheme of resolved momentum control applies dynamic constraints to track reference trajectories in total linear/angular momentum [6]. This is used to generate whole body motions (e.g., walking) for humanoid robots, whereas the controlled reduction approach exploits symmetry-based constraints in joint momenta to reduce the dimensionality of the trajectory generation problem.

This paper presents a first step towards the application of geometric controlled reduction to real robots by casting this framework into desired accelerations for inverse dynamics control. We choose a priori to enforce $k$ nonholonomic constraints that correspond to the momentum conservation laws required to perform a $k$-stage controlled reduction. By formulating the constraints in acceleration space, as was proposed in [14], and by carefully choosing the nullspace projector of those constraints, we construct desired accelerations that provide constraint enforcement and orthogonal decoupling of reference accelerations in the robot’s sagittal
plane. This similarly preserves the provable stability properties of the \( k \) divided coordinates from [2]. We can then insert any stable dynamics in the decoupled sagittal-plane subsystem to provide stability of the full-order dynamics.

The resulting control law is surprisingly simple. It depends on a trivial upper-triangular \( k \times k \) matrix inverse \textit{without} nesting any energy-shaping control terms to enforce the constraints for each stage of reduction. We also find that adding a simple linear proportional controller will linearize the output dynamics to render the constraint surface exponentially attractive in the phase space. The entire controller is easier to implement in practice, yet can be shown equivalent to the corresponding energy-shaping controller when the constraints are enforced. This also avoids some of the robustness problems often found with linearized constraints (e.g., bipedal walkers with linearized yaw/lean dynamics are very sensitive to steering perturbations [5]).

We demonstrate the effectiveness of this approach with numerical simulations. We first reproduce walking gaits for a 5-DOF dynamic walking biped in 3D space, showing correspondence to gaits produced with energy-shaping control in [4]. We then present preliminary simulation results for a realistic lower-extremity humanoid robot with 16 DOFs to show that the method easily scales to highly-redundant humanoid robots, opening the way to practical applications.

II. Biped Dynamics

We now construct a biped model that is simple enough to be analytically tractable, which we will use to illustrate the controlled reduction framework in Section III. The biped of Fig. 1 has five DOFs with a hip, splayed legs, and a torso. Although this model is a 3D extension of the 2D compass-gait-with-torso biped, the 3D biped does not have stable passive walking gaits down shallow slopes. We will therefore design control laws that exploit the existence of passive gaits in the sagittal plane to render this 3D biped pseudo-passively stable. This will be validated by simulation in Section V.

We represent the configuration space of the 5-DOF biped by \( \mathbb{Q} = \mathbb{T}^2 \times \mathbb{T}^3 \) with coordinates \( q = (\psi, \varphi, \theta)^T \), where \( \psi \) is the yaw (or heading), \( \varphi \) is the roll (or lean) from vertical, and \( \theta = (\theta_s, \theta_t, \theta_m)^T \) is the vector of sagittal-plane (pitch) variables as in the 2D model. The dynamical system state is given by \( x = (q^T, \dot{q}^T)^T \) in phase space\(^1 \mathbb{T} \mathbb{Q} \). All other parameters, including leg splay angle \( \rho \), are held constant.

We show how the continuous single-support dynamics are derived using Lagrangian mechanics, but defer symbolic expressions for modeling terms to [2]. To begin, the scalar Lagrangian function is defined as the difference between the robot’s kinetic and potential energies:

\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - \mathcal{V}(q),
\]

where \( 5 \times 5 \) inertia matrix \( M \) has the form

\[
M(\varphi, \theta) = \begin{pmatrix}
    m_\varphi(\varphi, \theta) & M_{\varphi \theta}(\varphi, \theta) \\
    M_{\varphi \theta}^T(\varphi, \theta) & M_{\theta}(\theta)
\end{pmatrix},
\]

where \( M_\theta \) is the \( 3 \times 3 \) inertia submatrix corresponding to the sagittal-plane DOFs. Note that \( M \) does not depend on \( \psi \), so yaw is called a cyclic variable. We see that the lower-right \( 4 \times 4 \) submatrix only depends on \( \theta \) and thus has another cyclic variable \( \varphi \). This recursively cyclic structure (a form of rotational symmetry) is shown to be a general property of open kinematic chains in [2], [4], which will be important to our control law design. We note that the potential energy has a similar cyclic form [2].

The Lagrangian \( L \) is used to compute the Euler-Lagrange (E-L) equations giving the 2nd-order dynamics of the controlled robot:

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + N(q) = Bu, \tag{2}
\]

where \( 5 \times 5 \) matrix \( C \) contains the Coriolis/centrifugal terms, \( N = \nabla_q \mathcal{V} \) is the gravitational force vector, and \( 5 \times 5 \) matrix \( B \) maps control inputs \( u \) to the generalized torques \( \tau \) by

\[
B = \begin{pmatrix}
    I_{2 \times 2} & 0_{2 \times 3} \\
    0_{3 \times 2} & B_\theta
\end{pmatrix}
\]

with planar map \( B_\theta \in \mathbb{R}^{3 \times 3} \).

Since walking involves discontinuous impact events, system (2) represents the single-support phase of the biped’s hybrid system dynamics. Impact events are triggered by guard set \( G \) of states where the swing foot is in contact with ground, and the instantaneous reset map \( \Delta(q, \dot{q}) \) is computed following the method of [15]. The biped dynamics are then fully characterized by the hybrid system

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + N(q) = Bu \quad \text{for } x \notin G
\]

\[
(q(t^+), \dot{q}(t^+)) = \Delta(q(t^-), \dot{q}(t^-)) \quad \text{for } x \in G
\]

The signs of \( w \) and \( \rho \) are also flipped at impact to model the change in stance leg. We now revisit the reduction-based control method, which we seek to improve for this biped.

III. Controlled Constraints and Reduction

Geometric reduction requires the existence of symmetry in a mechanical system’s dynamics, which induces an invariant surface in phase space \( T \mathbb{Q} \). Symmetries are often found in the form of conservation laws, where a physical quantity of the system is conserved by the dynamics. Therefore, these conservation laws can be expressed as nonholonomic constraints of the first-order form \( J_c(q) \dot{q} = b(q) \) or second-order form \( J_c(q) \ddot{q} = b(q, \dot{q}) \) (note the former case induces

\[\text{Fig. 1. The sagittal and frontal planes of a 3D bipedal torso robot and controlled reduction overview: the first reduction stage divides out the yaw DOF of the transverse plane, and the second stage divides out the lean DOF of the frontal plane, yielding the dynamics of the planar biped.}\]
the latter case). If \( J_c \) has row rank \( k \) in the former case, then the system dynamics restricted to the invariant surface

\[ \mathcal{Z} = \{ (q, \dot{q}) | J_c(q) \dot{q} = b(q) \} \]  

(3)
give an \((n - k)\)-DOF reduced-order system. The constraints then uniquely describe the “divided” coordinates, \( q_c \), based on the reduced coordinates \( q_c \), where \( q = (q_c^T, q_i^T)^T \).

However, naturally occurring symmetries usually do not provide stability in these divided coordinates (e.g., rotational symmetry for a biped’s heading). It was proposed in [2], [5] that control be used to shape momentum conservation laws in a manner that yields stability in the divided DOFs as well as meaningful reduced-order dynamics. In our biped example, the yaw and lean DOFs will be divided to yield a decoupled planar subsystem, for which control law design is substantially simpler and can exploit passive dynamics. This process of reduction-based control is depicted in Fig. 1. We now discuss how these controlled nonholonomic constraints are chosen to provide stability for these divided coordinates.

A. Stability via Controlled Momentum Constraints

The works [2], [4], [5] design a control method based on the classical theory of Routhian reduction, which exploits conservation of generalized momentum arising from the existence of cyclic variables. Although this momentum quantity is typically defined as \( p = M \dot{q} \), we exploit the recursively cyclic structure of the rigid-body inertia matrix \( M \) by instead defining the generalized momentum \( p = \hat{M} \dot{q} \) with respect to matrix \( \hat{M} \), defined by upper-triangular blocks from \( M \):

\[
\hat{M}(q) = \begin{pmatrix}
\hat{M}_1(q_c, q_r) & \hat{M}_{12}(q_c, q_r) \\
0 & \hat{M}_2(q_r)
\end{pmatrix},
\]

where \( \hat{M}_1 \in \mathbb{R}^{k \times k} \) corresponds to the upper-triangular part of the top-left \( k \times k \) submatrix in \( M \), and \( \hat{M}_{12} \in \mathbb{R}^{k \times (n-k)} \), \( \hat{M}_2 \in \mathbb{R}^{(n-k) \times (n-k)} \) are, respectively, the top-right and bottom-right submatrices in \( M \).

In terms of the 5-DOF biped model, we have the configuration partition \( q_c = (\psi, \phi)^T \) and \( q_r = \theta \) for a two-stage controlled reduction for yaw and lean. We would then write

\[
\hat{M}_1(q) = \begin{pmatrix}
m_{\psi}(\varphi, \theta) & m_{\psi\varphi}(\varphi, \theta) \\
0 & m_{\varphi}(\theta)
\end{pmatrix},
\]

\[
\hat{M}_{12}(q) = \begin{pmatrix}
m_{\psi\theta}(\varphi, \theta) \\
M_{\psi\theta}(\varphi, \theta)
\end{pmatrix},
\]

\[
\hat{M}_2(q_r) = M_{\theta}(\theta).
\]

Given these definitions, we wish to create a 2\((n-k)\)-dimensional zero dynamics defined by controlled momentum constraints that stabilize coordinates \( q_c \) [2]:

\[
p_c := \begin{bmatrix}
I_{k \times k} & 0_{k \times (n-k)}
\end{bmatrix} \hat{M} \dot{q} = -K(q_c - \bar{q}_c)
\]

\[
\Leftrightarrow \begin{bmatrix}
\hat{M}_1 & M_{12}
\end{bmatrix} \dot{q} = -K(q_c - \bar{q}_c)
\]

\[
\Leftrightarrow \ddot{q}_c = -K^{-1} \left[ K(q_c - \bar{q}_c) + M_{12} \bar{q}_c \right],
\]

(4)

where \( K \in \mathbb{R}^{k \times k} \) is a diagonal positive-definite matrix of constant gains, and \( \bar{q}_c \in \mathbb{T}^k \) is a piecewise-constant vector of desired set-points for coordinates \( q_c \). This defines the invariant surface \( \mathcal{Z}_{\bar{q}_c} \), associated with constraint (4) as in (3), where \( J_c = [M_1, M_{12}] \) and \( b = -K(q_c - \bar{q}_c) \) is continuously parameterized by desired set-point \( \bar{q}_c \).

Due to the recursively cyclic and upper-triangular structure of \( M \), it is easily shown that scaling matrices \( M^{-1}_1 K \) and \( M_{12}^{-1} M_{12} \) have no dependence on configuration elements \( q_{1,..,i} \) in row \( i \) (where \( i \in 1,..,k \)). We then see that equation (4) represents a homogeneous first-order linear system in \( q_c \) with time-varying coefficients based on trajectories \( (q_c(t), q_r(t)) \). Moreover, \( M_{12}^{-1} K \) is positive-definite by the positive-definite property of inertia matrix \( M \) and consequently its diagonal blocks. Therefore, system (4) has negative gain linearity in \( q_c \), by which asymptotic convergence of its trajectories can be proven as in [2]:

**Lemma 1:** If reduced trajectories \((q_c(t), \dot{q}_r(t))\) are asymptotically convergent to \( T \)-periodic, bounded solution \((\bar{q}_c^*(t), \dot{\bar{q}}^*_r(t))\), trajectories of system (4) are asymptotically convergent to unique, \( T \)-periodic, and bounded solution \( (\dot{q}_c(t), \dot{q}_r(t)) \) in a neighborhood about \( \bar{q}_c \). Therefore, full-order trajectories \((q(t), \dot{q}(t))\) are asymptotically convergent to \( T \)-periodic, bounded solution \((\dot{q}_c^*(t), \dot{q}_r^*(t))\).

Asymptotic stability in the reduced subsystem of coordinates \((q_c, \dot{q}_r)\) then implies asymptotic stability in first-order system (4) for divided coordinates \( q_c \). The control problem is thus reduced to the design of an asymptotically stable limit cycle for the reduced coordinates \((q_c, \dot{q}_r)\). In this fashion, full-order asymptotically stable limit cycles can be constructed for locomotor patterns.

B. Sagittal-Plane Decoupling via Controlled Reduction

Now we render the reduced dynamics (e.g., the sagittal plane) decoupled from the coordinates constrained by (4). For the sake of clarity, we present the decoupling reduction for the particular case of the 5-DOF biped model with two constraints. Note, however, that in general this method applies to an \( n \)-DOF robot with \( k \) constraints.

We wish to design a control law such that the restriction of the closed-loop system (2) to surface \( \mathcal{Z}_{\bar{q}_c} \) is given by the reduced-order planar system

\[
M_{\theta}(\theta) \dot{\theta} + C_{\theta}(\theta, \dot{\theta}) \dot{\theta} + N_{\theta}(\theta) = B_{\theta} v_{\theta}
\]

(5)

Here, \( C_{\theta} \in \mathbb{R}^{3 \times 3} \) contains the Coriolis terms from \( M_{\theta}, N_{\theta} \in \mathbb{R}^{3} \) is a gravitational vector, and \( v_{\theta} \in \mathbb{R}^{3} \) is an auxiliary control input.

An energy-shaping control law is designed in [5] to transform full-order Lagrangian \( \mathcal{L} \) into a so-called almost-cyclic Lagrangian \( \mathcal{L}_\lambda \), defined in coordinates as

\[
\mathcal{L}_\lambda(q, \dot{q}) := \mathcal{L}(\varphi, \theta, \dot{q}) + L_{\lambda}^{\text{mg}}(q, \dot{q})
\]

(6)

where the shaped inertia has two Schur complements of \( M \):

\[
M_{\lambda} = M - \begin{pmatrix}
0 & M_{c\psi}^T \bar{M}_{\psi} \bar{M}_{\psi \psi}^{-1} M_{\psi \psi} \bar{M}_{\psi}^T \\
0 & M_{c\psi}^T \bar{M}_{\psi} \bar{M}_{\psi \psi}^{-1} M_{\psi \psi} \bar{M}_{\psi}^T
\end{pmatrix}
\]

and gyroscopic term \( Q_{\lambda} \) and potential energy \( V_{\lambda} \) depend on the cyclic variables through function \( \lambda(q_c) = -K(q_c - \bar{q}_c) \). We defer the detailed expressions for these terms to [2], [5].

**Remark 1:** An important property of \( \mathcal{L}_\lambda \) is that the associated E-L equations render \( \mathcal{Z}_{\bar{q}_c} \) invariant [2] (i.e., it ensures that the constraints (4) will be satisfied for all time given they are satisfied initially). In order to account for initial
conditions outside of $Z_{\bar{q}_c}$, we will design an auxiliary controller in Section IV that further renders this surface globally attractive. This will also correct for constraint violations when changing the biped’s desired heading.

**Proposition 1:** The closed-loop Euler-Lagrange equations of $L_\lambda$ restricted to $Z_{\bar{q}_c}$ are given by planar system (5).

The proof is given in [2], [5]. I.e., geometric reduction of the closed-loop system with respect to controlled conservation law (4) projects onto decoupled planar system (5). However, the control law imposing this reduction is very complex and difficult to implement in practice. The Lagrangian-shaping torques are given by inverting the dynamics and reinserting the original dynamics plus augmenting terms:

$$\tau_\lambda := C\ddot{q} + N - MM_\lambda^{-1}(CM_\lambda\ddot{q} + CQ_\lambda, \ddot{q} + N_\lambda - v),$$

(7)

where $v$ is an auxiliary controller rendering $Z_{\bar{q}_c}$ globally exponentially attractive. In particular, matrix $M_\lambda$ must be recursively constructed and will be high dimensional for a highly-redundant humanoid robot. The inverse of this matrix is not only demanding computationally, but also risks exploding any modeling errors in the original $M$ matrix. Moreover, this control law requires explicit computation of shaped Coriolis and gyroscopic terms $CM_\lambda$ and $CQ_\lambda$, respectively, in which each row of conservation function $\lambda(q_c)$ contributes nonlinear terms needed for each stage of reduction. Therefore, the symbolic complexity of the controller increases exponentially with each stage of reduction.

In the next section, we circumvent some of these issues (while keeping the strong stability results) by designing an acceleration-based controller using orthogonal projections.

### IV. Acceleration-Based Control

We now present the main result of this paper. We reformulate controlled reduction by computing desired accelerations that will enforce the momentum constraints of (4) and decouple the reduced coordinates $(q_r, \dot{q}_r)$. We will have by construction a controller that will enforce the theoretical results from geometric controlled reduction. This desired acceleration can then be achieved using standard inverse dynamics control.

We consider the general case of an $n$-DOF robot with $k$ first-order constraints of the form

$$J_c(q)\ddot{q} = b(q).$$

(8)

In order to impose a $k$-stage decoupling reduction, we define $J_c = [M_1, M_{12}]$ and $b = -K(q_c - \ddot{q}_c)$ as in Section III-A.

#### A. Constrained Accelerations for Dynamical Decoupling

We proceed in two main steps. We first rewrite the constraints in joint acceleration space, following the ideas from [14]. We then strategically choose a generalized inverse of constraint Jacobian $J_c$ in order to compute a solution for desired accelerations that maintains the constraints and decouples the unconstrained dynamics.

First-order constraint Jacobian $J_c$ maps joint velocities to momenta in (8), but this Jacobian can also map joint accelerations to torques. Therefore, we take the time-derivative of (8) to obtain the second-order constraint

$$J_c\dddot{q} = -J_c\ddot{q} + b,$$

(9)

where $\dddot{J}_c = [\dddot{M}_1, \dddot{M}_{12}]$ and $\dddot{b} = -K\dddot{q}_c$.

We wish to derive a desired acceleration $\dddot{q_d} \in \mathbb{R}^n$ that enforces this constraint and follows a reference acceleration $\dddot{q}_{ref} = (\dddot{q}_{ref}, \dddot{q}_{ref})^T \in \mathbb{R}^n$ within the constraint nullspace. All solutions for this desired acceleration can be given by

$$\dddot{q_d} = J_c^T(-\dddot{J}_c\ddot{q} - K\dddot{q}_c) + (I - J_c^TJ_c)\dddot{q}_{ref},$$

(10)

where $J_c^T$ denotes any generalized inverse of $J_c$, i.e., a matrix such that $J_c^TJ_c = J_c$. Given such an inverse, the previous equation gives the possible solutions (if any) that will enforce the constraints. By designing our momenta/inertial constraints based on the robot’s natural dynamics, we know that our constraints are feasible and a solution will exist (e.g., the controlled reduction result of Section III).

Since $J_c$ is full row rank, we can choose an inverse of the form $J_c^T = WJ_c^T(J_cWJ_c^T)^{-1}$, where $W$ is a positive definite weight matrix. Now we can manipulate how the $\dddot{q}_{ref}$ accelerations will be projected in the null space of the constraints. We choose $W = M^{-T}$ to find that

$$J_c^T = \begin{pmatrix} M_1^{-1} \\ 0 \end{pmatrix},$$

(11)

and therefore the nullspace projector takes the simple form

$$(I - J_c^TJ_c) = \begin{pmatrix} 0 & -M_1^{-1}M_{12} \\ 0 & I \end{pmatrix}. $$

(12)

This choice of weight matrix (and nullspace projector) renders orthogonal the projections of the constrained and unconstrained dynamics (recall that $J_c$ is defined by $M$, and $MW = WM = I$). We can now express desired accelerations (10) in terms of the partitions:

$$\dddot{q_d} = \begin{pmatrix} -M_1^{-1}(\dddot{J}_c\ddot{q} + K\dddot{q}_c + M_{12}\dddot{q}_{rd}) \\ \dddot{q}_{rd} \end{pmatrix},$$

(13)

where nullspace projector (12) removes any dependence on $\dddot{q}_{rd}$, i.e., the first $k$ coordinates evolve according to the constraints rather than a reference trajectory), leaving command over the unconstrained reference acceleration $\dddot{q}_{rd}$. And, this term is decoupled in the unconstrained partition.

In order to impose joint accelerations (13), we define the inverse dynamics controller:

$$\tau(\ddot{q}, \dot{q}) := M(\dddot{q}_d + \begin{pmatrix} v \\ 0 \end{pmatrix}) + C\ddot{q} + N,$$

(14)

where $v$ is an auxiliary control term in the constrained dynamics that will be derived to render $Z_{\bar{q}_c}$ exponentially attractive so that first-order constraint (8) for a specific $\dddot{q}_c$ holds in addition to second-order (9). Applying control law (14) to system (2), the closed-loop dynamics can finally be decomposed into orthogonal subsystems:

$$\dddot{q}_c = -M_1^{-1}(\dddot{J}_c\ddot{q} + K\dddot{q}_c + M_{12}\dddot{q}_{rd}) + v$$

$$\dddot{q}_r = \dddot{q}_{rd}. $$

(15)

**Remark 2:** We see from system (15) that if momentum constraints are defined for every out-of-plane DOF (i.e., yaw and lean), the reduced accelerations $\dddot{q}_r$ correspond to the sagittal plane and are decoupled from the constraint satisfaction accelerations. We can now design $\dddot{q}_{rd}$ such that the unconstrained system has any desired dynamics.
example, we can close a feedback loop with \((q_r, \dot{q}_r)\) to re-insert the planar subsystem dynamics with energy-shaping terms to produce pseudo-passive walking as in [5]. Or, we can impose a designed reference trajectory for the sagittal plane, allowing lower-dimensional planning for strategies related to the center of pressure (e.g., Zero Moment Point).

**Remark 3:** If we return to the 5-DOF biped example and choose \(\dot{q}_{r_d}\) to be the accelerations of the planar subsystem dynamics (5), then by construction the restriction of closed-loop system (19) is equivalent to the restricted system associated with almost-cyclic Lagrangian (6). I.e., reduction-based control (7) is equivalent to (14) on the surface \(Z_{\bar{q}_c}\). We will see, however, that the two control methods behave differently off this surface. The energy-shaping terms in (7) have different meaning when first-order constraint (8) is violated – in particular, sagittal-plane decoupling is lost.

We will see in Section V that the biped’s discontinuous impact events tend to violate constraint (8), i.e., surface \(Z_{\bar{q}_c}\) is not hybrid invariant in impulsive system (3). We now design a controller for input \(v\) rendering \(Z_{\bar{q}_c}\) globally exponentially attractive, in order to enforce the constraints shortly after each impact event.

**B. Rendering \(Z_{\bar{q}_c}\) Exponentially Attractive**

Although second-order constraint (9) is always enforced under control law (14), this implies one of infinitely-many first-order constraints within a constant offset from desired (8). In order to stabilize periodic orbits in \(q_c\) about a specific set-point \(\bar{q}_c\), we use auxiliary control \(v\) to render the corresponding surface \(Z_{\bar{q}_c}\) globally exponentially attractive. This is equivalent to zeroing \(k\) outputs of the vector form \(\dot{y} := J_c \dot{q} - b\), so we wish to linearize the associated output dynamics into the exponentially stable system

\[
\dot{y} = -LY_y, \tag{16}
\]

for some positive-definite gain matrix \(L \in \mathbb{R}^{k \times k}\). Rewriting in terms of the constraints we have the equivalent formulation

\[
\dot{J}_c \dot{q} + J_d \dot{q} - \bar{b} = -L(J_c \dot{q} - b). \tag{17}
\]

Plugging (15) into \(\dot{q}\), we solve for the linearizing control law:

\[
v(q_c, \dot{q}) = -\tilde{M}_1^{-1}L(J_c \dot{q} - b). \tag{18}
\]

**Remark 4:** This is a proportional controller in the first-order constraints. It is somewhat remarkable that this would globally stabilize the differential outputs without any dependence on the unconstrained system, but this is a natural result of the orthogonally decomposed dynamics (15). This enables a simplified expression for the output linearizing control, which must otherwise be derived from complicated Lie derivatives involving the full inertia matrix inverse (e.g., [2], [4], [5]). Note that this input is zero when restricted to the constraint surface, i.e., \(v_{|Z_{\bar{q}_c}} = 0\).

We can now write the closed-loop accelerations that render \(Z_{\bar{q}_c}\) invariant and exponentially attractive as

\[
\begin{align*}
\ddot{q}_c &= -\tilde{M}_1^{-1}(J_c \ddot{q} + Kq_c + M_{12}\ddot{q}_{r_d} + L(J_c \dot{q} - b)) \\
\ddot{q}_r &= \ddot{q}_{r_d}. \tag{19}
\end{align*}
\]

Aside from the time-derivative of \(J_c\), this closed-loop system and the underlying control law (14) have a noticeably simple form. The only inverse required is that of a positive-definite, upper-triangular matrix \(M_1\), which is trivial to compute and therefore realistic for implementation on a robotic platform.

**Remark 5:** This controller can be interpreted as a special case of operational space control [11], where tracking \(\dot{q}_r\) is the operational task and constraint enforcement corresponds to the nullspace dynamics. By imposing these constraints, we guarantee asymptotic stability in the nullspace of the task, which is generally difficult to prove. The pseudo-inverse that we use differs from the traditional inertia weighted inverse [8] for dynamic decoupling because we impose generalized momentum constraints that are non-conventional (see Sect. III-A). A detailed comparison of these two approaches would be interesting but is outside the scope of this paper.

**V. NUMERICAL APPLICATIONS**

We now show that the new controller (14) performs similarly to reduction-based controller (7) on the 5-DOF biped defined in Section II. We then present preliminary simulation results of a 16-DOF biped balancing on one foot, showing that the new framework can scale to highly-redundant humanoid robots.

**A. 5-DOF Bipedal Robot**

We adopt the above acceleration-based framework with yaw and lean constraints defined by (4). Recall the 5-DOF configuration vector is partitioned into constrained coordinates \(q_c = (\psi, \phi)^T\) and unconstrained coordinates \(q_r = \theta\). In order to create pseudo-passive walking gaits, we close an outer feedback loop around decoupled system (19), inserting sagittal dynamics (5) into the unconstrained accelerations:

\[
\ddot{\theta}_d := -M_{\theta}^{-1}(\theta) \left(C_{\theta}(\theta, \dot{\theta})\dot{\theta} + N_{\theta}(\theta) - B_{\theta}\nu_{\theta}\right). \tag{20}
\]

Hence, system (19) can be controlled as a planar 3-DOF biped with well-known passivity-based techniques in \(\nu_{\theta}\).

1) **Planar subsystem controller:** The decoupled sagittal-plane subsystem is controlled using the same passivity-based controller of [4]. In particular, we employ slope-changing “controlled symmetries,” a method that shapes the potential energy to impose symmetries on the system dynamics with respect to ground orientation. Just as the planar compass-gait biped has known passive limit cycles down shallow slopes, so does the compass-gait-with-torso biped. These known gaits are harnessed by virtually rotating the gravity vector to mimic downhill passive dynamics on flat ground.
In addition, a PD control law is needed to upright the torso, which behaves as an unstable inverted pendulum, during otherwise passive walking gaits [13]. The subsystem controller that both uprights the torso and maps passive gaits in this manner is given by

\[ v_\theta = B_\theta^{-1} \left( N_\theta(\theta) - N_\theta(\beta + \theta) + (0, v_{pd}, 0)^T \right) \]  

where \( v_{pd} = -k_p(\theta_1 + \beta) - k_s\dot{\theta}_1 \) and \( \beta = 0.052 \) rad, the slope angle yielding the desired passive limit cycle.

### Simulation results

We simulate straight-ahead walking about heading \( \psi = 0 \) using both control laws (14) and (7), comparing the torque profiles in Fig. 2 and 2-step periodic limit cycles in Fig. 3. We see in Fig. 2 that the control laws differ immediately after impact due to the discontinuous jump off surface \( z_{q_k} \), resulting in the slightly different limit cycles of Fig. 3. As we would expect, the input torques from the two different control laws converge mid-step cycle, indicating that subcontroller (18) has driven the state to surface \( z_{q_k} \). The two control laws are then equivalent until the next impact event introduces another discontinuous jump.

In order to show local exponential stability of the 2-periodic gait, we define a discrete map between ever other impact event and find the fixed-points \( x_1^* \) and \( x_3^* \) of the limit cycles from controllers (14) and (7), respectively. This map can be numerically linearized about these fixed-points to compute their approximate eigenvalues. We find that the eigenvalues are strictly within the unit circle, thus confirming local exponential stability of the corresponding limit cycles.

In order to evaluate gait efficiency, we integrate \( \dot{q}^2 u \) under acceleration-based controller (14) to obtain the net work per step, finding that the specific average mechanical power is 0.267 W/kg. Moreover, the specific mechanical cost of transport (work done per unit weight per unit distance) for this gait is \( c_{nt} = 0.039 \), which is almost identical to the cost under reduction-based controller (7). This compares favorably against popular walking robots such as the Cornell biped at \( c_{nt} = 0.055 \) and Honda ASIMO at \( c_{nt} = 1.6 \) [1].

We can also command the heading of the biped by setting desired yaw \( \psi \) at the beginning of every 2-step cycle. This does not destabilize the biped so long as the set-point is changed in small increments (this causes additional impact error from new desired surface \( z_{q_k} \)). One such steering simulation can be seen in the attached movie. These steering gaits enable path planning for 3D dynamic walkers based on asymptotically stable gait primitives from [3].

### 16-DOF Humanoid Robot

We now use the new controller on a more realistic model. We use a physical simulation of a 16-DOF robot that is modeled as the lower body of the Sarcos Humanoid robot (i.e., we do not include the arms and head of the humanoid for simplicity). We integrate the desired accelerations in a full floating-base inverse dynamics controller developed in [10] that was already successfully used on real robotic platforms.

We impose momentum constraints on all non-sagittal DOFs as well as two DOFs at the torso, resulting in a total of 10 constraints. The reduced system has 6 DOFs including the hip, knee, and ankle flexion/extension of each leg. We can then design accelerations \( \ddot{q} \) in this reduced system to track desired joint trajectories \( q \) using a simple PD controller. Hence, we have reduced the dimensionality of the trajectory-planning problem to less than half the dimension of the full system. However, it is still not obvious how the sagittal-plane dynamics of a humanoid should be designed for efficient and robust bipedal locomotion, as these subsystems generally do not have passive gaits to be exploited. We are currently developing such trajectories, but in lieu of walking we offer a preliminary experiment of balancing on one foot.

We produce periodic motions in the sagittal plane, starting at 0.25Hz and increasing to 1Hz. We change these patterns to show that the constrained DOFs are asymptotically stable (they converge to a periodic orbit for each sagittal pattern). Results of the experiments can be seen in Fig. 4 and the attached movie. We designed the trajectories such that during the 1Hz motion the sagittal-plane dynamics move the center of pressure to the front and back limits of the foot (Fig. V-B). It is interesting to note that despite these movements at the limit of stability, the robot stays stable in the lateral direction. It is also worth mentioning that performing these sagittal-plane movements while using high gain PD control to fix the positions of the other DOFs would lead to a fall.

### VI. Conclusions

The main contribution of this paper is the reformulation of controlled reduction from [2]–[5] into a control framework of constrained accelerations and inverse dynamics. The resulting control law is surprisingly simple and therefore is an important first step toward applying controlled reduction (and energy shaping in general) to humanoid robotic platforms.

This controller imposes momentum conservation laws to stabilize a set of DOFs around desired set-points, reducing the control problem to a meaningful lower-dimensional mechanical system. It is then possible to design trajectories and a controller for the reduced subsystem that is decoupled from the other DOFs. Asymptotic stability of the complete system is guaranteed by asymptotic stability of the reduced system.

In the context of locomotion, this reduces the dimensionality of pattern generators yet still provides stable gaits for the entire system. We applied this framework to bipedal robots, for which we decoupled the sagittal-plane dynamics for periodic motion generation. Numerical simulations of walking gaits of a 5-DOF biped robot in 3D space confirmed correspondence between the original controlled reduction framework and the new control method. This enables pseudo-passive bipedal walking that can be stably steered through complex 3D environments as in [3]. We also presented preliminary results on decoupled balancing for a 16-DOF robot, showing that the new controller scales to a realistic number of DOFs for a humanoid robot. This has not been shown possible with the original control law.

This contribution opens several paths of research. The next step is to implement both pseudo-passive and trajectory-based walking gaits (planned in the sagittal plane) on a real humanoid platform. This paper also establishes an interesting bridge between controlled geometric reduction and acceleration-based controllers. This inspires further developments in energy-efficient torque controllers for humanoid robots by better understanding the commonalities between these approaches.
Fig. 3. Phase portrait (left) and joint positions over time (right) comparing straight-ahead limit cycles from acceleration-based control (solid lines) and reduction-based control (dotted lines) from [4].

5-DOF Physical Parameters : $M_l = 10$ kg, $t_l = 0.5$ m, $M_b = 10$ kg, $w = 0.1$ m, $m = 5$ kg, $l = 1$ m, $\rho = 0.0188$ rad

5-DOF Control Parameters : $K_1 = 20$, $K_2 = 30$, $L_1 = 30$, $L_2 = 15$, $k_p = 700$, $k_d = 200$, $\beta = 0.052$ rad, $U_{\text{max}} = 30$ Nm

$x_1^L \approx (0.0572, 0.0060, -0.2679, 0.0040, 0.2693, 0.0678, -0.0179, -1.3313, 0.0623, -1.5930)^T$  

$x_2^L \approx (0.0544, 0.0062, -0.2543, 0.0201, 0.2558, 0.0673, -0.0173, -1.2871, 0.0673, -1.7233)^T$

---

Fig. 4. (a) Trajectories of the sagittal movements of the knees and hips. Every 8 seconds we change the movement, increasing the frequency from 0.25Hz to 1Hz. (b) Trajectory of constrained DOFs. We notice the asymptotic convergence to periodic trajectories for each change of movement of the sagittal-plane DOFs. The horizontal dotted lines show the set points $\dot{q}_i$. The right leg is plotted in black and the left in gray. The vertical lines show the start of new periodic movements in the sagittal-plane. (c) Snapshot of 1Hz movements. Note the position of the CoP at the limits of the foot (red ball).

---

REFERENCES


