Notes on The Cramér-Rao Bound

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1 The Cramér-Rao Lower Bound Theorem

1.1 Single Parameter Continuous Case

Theorem 1. *Let* $\mathbf{X} = \{\mathbf{x}_i\}_{i=1}^N$ *have a joint density function depending upon a single parameter* θ *such that the likelihood is* $p(\mathbf{X};\theta)$. *Any function* $T(\mathbf{X})$ *which is an unbiased estimator of* $\phi = g(\theta)$ *has variance bounded below by a uniform bound (called the Cramér-Rao lower bound and denoted by CRLB) namely:*

$$Var(T(\mathbf{X})) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{\left\langle -\frac{\partial^2 \log p(\mathbf{X};\theta)}{\partial \theta^2} \right\rangle} \tag{1}$$

Where the expectation is carried out over all possible datasets $\mathbf{X}$.

Proof. The proof uses the result that for any two random variables U and V, the correlation coefficient $\rho(U,V)$ satisfies $-1 \leq \rho(U,V) \leq 1$, where $\rho(U,V)$ is defined as follows:

$$\rho^2(U,V) = \frac{(\mathrm{Cov}(U,V))^2}{\mathrm{Var}(U)\mathrm{Var}(V)} \leq 1 \tag{2}$$

Assume $U = T(\mathbf{X})$, and $V = \partial \log p(\mathbf{X};\theta)/\partial\theta$. The proof follows in 3 stages:

1. Prove that $\langle V \rangle = 0$
2. Prove that $\mathrm{Cov}(U,V) = \partial g(\theta)/\partial\theta$
3. Prove that $\mathrm{Var}(V) = \left\langle -\partial^2 \log p(\mathbf{X};\theta)/\partial\theta^2 \right\rangle$

We now carry out the proof as follows:

1. Prove that $\langle V \rangle = 0$

$$\left\langle \frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right\rangle = \int \frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} p(\mathbf{X};\theta) d\mathbf{X} = \int \frac{\partial p(\mathbf{X};\theta)}{\partial\theta} d\mathbf{X} = \frac{\partial}{\partial\theta}\int p(\mathbf{X};\theta) d\mathbf{X} \tag{3}$$

$$= 0 \tag{4}$$

2. Prove that $\mathrm{Cov}(U,V) = \partial g(\theta)/\partial\theta$

$$\mathrm{Cov}(U,V) = \left\langle (T(\mathbf{X}) - g(\theta)) \left(\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right) \right\rangle \tag{5}$$
$$= \int (T(\mathbf{X}) - g(\theta)) \left(\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right) p(\mathbf{X};\theta) d\mathbf{X} \tag{6}$$
$$= \int (T(\mathbf{X}) - g(\theta)) \left(\frac{\partial p(\mathbf{X};\theta)}{\partial\theta} \right) d\mathbf{X} \tag{7}$$
$$= \int T(\mathbf{X}) \left(\frac{\partial p(\mathbf{X};\theta)}{\partial\theta} \right) d\mathbf{X} - \int g(\theta) \left(\frac{\partial p(\mathbf{X};\theta)}{\partial\theta} \right) d\mathbf{X} \tag{8}$$
$$= \int \frac{\partial T(\mathbf{X}) p(\mathbf{X};\theta)}{\partial\theta} d\mathbf{X} - g(\theta) \int \frac{\partial p(\mathbf{X};\theta)}{\partial\theta} d\mathbf{X} \tag{9}$$
$$= \frac{\partial}{\partial\theta} \int T(\mathbf{X}) p(\mathbf{X};\theta) d\mathbf{X} - g(\theta) \frac{\partial}{\partial\theta} \int p(\mathbf{X};\theta) d\mathbf{X} \tag{10}$$
$$= \frac{\partial}{\partial\theta} \langle T(\mathbf{X}) \rangle - 0 \tag{11}$$
$$= \frac{\partial g(\theta)}{\partial\theta} \tag{12}$$

3. Prove that $\mathrm{Var}(V) = \left\langle -\partial^2 \log p(\mathbf{X};\theta)/\partial\theta^2 \right\rangle$

$$\mathrm{Var}(V) = \left\langle \left(\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right)^2 \right\rangle \tag{13}$$
$$= \int \left(\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right)^2 p(\mathbf{X};\theta) d\mathbf{X} \tag{14}$$
$$= \int \left(\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} \right) \left(\frac{\partial p(\mathbf{X};\theta)}{\partial\theta} \right) d\mathbf{X} \tag{15}$$

Using $\frac{\partial^2 p}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(p \frac{\partial \log p}{\partial\theta} \right) = \frac{\partial p}{\partial\theta}\frac{\partial \log p}{\partial\theta} + p \frac{\partial^2 \log p}{\partial\theta^2}$

$$= \int \frac{\partial^2 p(\mathbf{X};\theta)}{\partial\theta^2} d\mathbf{X} - \int \frac{\partial^2 \log p(\mathbf{X};\theta)}{\partial\theta^2} p(\mathbf{X};\theta) d\mathbf{X} \tag{16}$$
$$= \left\langle -\frac{\partial^2 \log p(\mathbf{X};\theta)}{\partial\theta^2} \right\rangle \tag{17}$$

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