Deriving a Marginal Student-t Distribution

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Assume that $x$ is a Gaussian distributed random variable with a known mean $\mu$ but unknown precision (inverse variance) $\psi$. This unknown precision is distributed according to a Gamma distribution with order and scale $a$ and $b$ respectively:

$$p(x|\psi) = \left(\frac{\psi}{2\pi}\right)^{1/2} \exp \left\{ -\frac{\psi}{2} (x - \mu)^2 \right\}$$

$$p(\psi) = \frac{b^a}{\Gamma(a)} \psi^{a-1} \exp(-b\psi)$$

The marginal distribution of $x$ is derived as follows:

$$p(x) = \int p(x|\psi)p(\psi)d\psi$$

$$= \frac{b^a}{\Gamma(a)} \left(\frac{1}{2\pi}\right)^{1/2} \int \psi^{a+1/2-1} \exp \left\{ - \left[ b + \frac{1}{2} (x - \mu)^2 \right] \psi \right\} d\psi$$

$$= \frac{b^a}{\Gamma(a)} \left(\frac{1}{2\pi}\right)^{1/2} \frac{\Gamma(a+1/2)}{\left[ b + \frac{1}{2} (x - \mu)^2 \right]^{a+1/2}}$$

$$= \frac{\Gamma(a+1/2)}{\Gamma(a)} \left(\frac{1}{2\pi}\right)^{1/2} \frac{b^a}{\left[ b + \frac{1}{2} (x - \mu)^2 \right]^{a+1/2}}$$

$$= \frac{\Gamma(a+1/2)}{\Gamma(a)} \left(\frac{1}{2\pi b}\right)^{1/2} \left[ 1 + \frac{(x - \mu)^2}{2b} \right]^{-(a+1/2)}$$

Hence the marginal distribution of $x$ is a Student-t distribution with shape parameter $a$ and scale parameter $b$. Figure 2 compares the conditional and marginal distributions of $x$ under several Gamma distributions.
Figure 1: For the given Gamma distribution over $\psi$, we can compare the conditional and marginal distributions. Note that a higher variance Gamma distribution causes a greater smearing, resulting in a heavier tailed marginal distribution.