Deriving a Marginal Student-t Distribution

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Assume that x is a Gaussian distributed random variable with a known mean μ but unknown precision (inverse variance) ψ. This unknown precision is distributed according to a Gamma distribution with order and scale a and b respectively:

$$\begin{aligned} p(x|\psi) &= \left(\frac{\psi}{2\pi}\right)^{1/2} \exp\left\{-\frac{\psi}{2}(x-\mu)^2\right\} \\ p(\psi) &= \frac{b^a}{\Gamma(a)}\psi^{a-1}\exp(-b\psi) \end{aligned}$$

The marginal distribution of x is derived as follows:

$$\begin{aligned} p(x) &= \int p(x|\psi)p(\psi)d\psi \\ &= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\int \psi^{a+1/2-1}\exp\left\{-\left[b+\frac{1}{2}(x-\mu)^2\right]\psi\right\}d\psi \\ &= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\frac{\Gamma(a+1/2)}{\left[b+\frac{1}{2}(x-\mu)^2\right]^{a+1/2}} \\ &= \frac{\Gamma(a+1/2)}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2}\frac{b^a}{\left[b+\frac{1}{2}(x-\mu)^2\right]^{a+1/2}} \\ &= \frac{\Gamma(a+1/2)}{\Gamma(a)}\left(\frac{1}{2\pi b}\right)^{1/2}\left[1+\frac{(x-\mu)^2}{2b}\right]^{-(a+1/2)} \end{aligned}$$

Hence the marginal distribution of x is a Student-t distribution with shape parameter a and scale parameter b. Figure 2 compares the conditional and marginal distributions of x under several Gamma distributions.

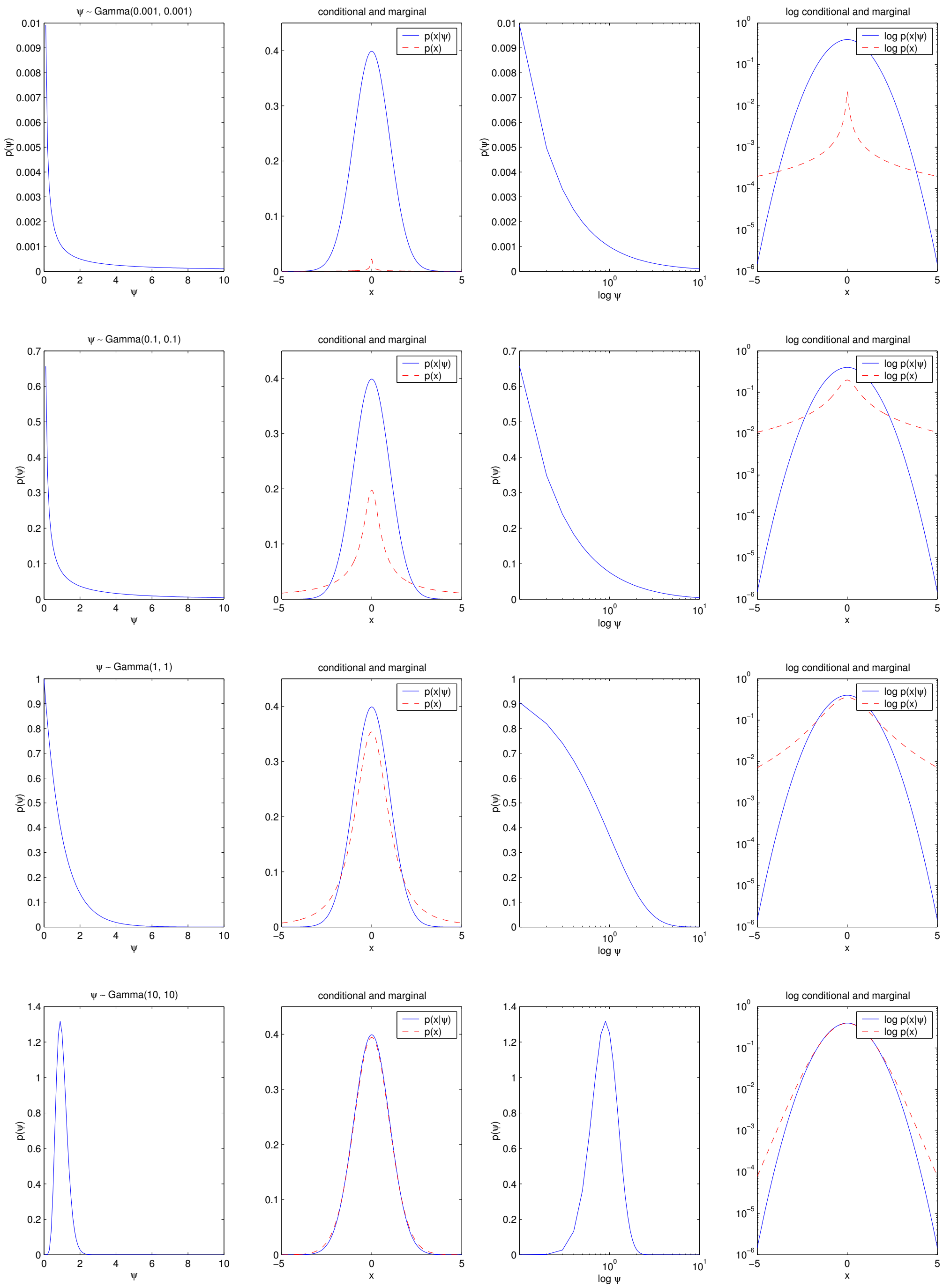

Figure 1: For the given Gamma distribution over ψ, we can compare the conditional and marginal distributions. Note that a higher variance Gamma distribution causes a greater *smearing*, resulting in a heavier tailed marginal distribution.