In the following problems, you should use MATLAB to compute numerical results and visualize the data, and Simulink for simulations. A handout about getting started with MATLAB is in

http://www-clmc.usc.edu/~cs545/CS545_homework.html

This web page also contains all the files needed below. IMPORTANT: In your solutions of the homework, also provide intermediate steps how you derived the solution to a problem.

1. (120 Points) A classical example in control theory is the cart-pole balancer (see figure).

The goal of the task is to apply a horizontal force $F$ such that the pole remains upright (i.e., $\theta = \pi/2$) and the cart stays at $x=0$, ideally such that the velocity states are zero, i.e., $\dot{\theta} = 0$, $\ddot{x} = 0$. Such a task is called a “regulator task” since the objective is to keep the controlled object at a particular setpoint. The second order equations of motion of the cart-pole system are:

$$
\ddot{x} = \frac{F}{m} - g \sin \theta \cos \theta + l \dot{\theta}^2 \cos \theta \quad \frac{M}{m} + (\cos \theta)^2
$$

$$
\ddot{\theta} = \frac{F}{m} \sin \theta - \frac{M + m}{m} g \cos \theta + \frac{l \dot{\theta}^2 \sin \theta \cos \theta}{l \left( \frac{M}{m} + (\cos \theta)^2 \right)}
$$

where $M$ denotes the mass of the cart, $m$ the mass of the pole, $g$ the gravity constant, and $l$ the distance of the pole center of mass from the pole hinge.

a) Reformulate the equations of motion as a set of first order differential equations, using $x = [\dot{x}, x, \dot{\theta}, \theta]^T$. 
b) Derive the equilibrium points of the system without the input, i.e., \( \dot{x} = Ax \).

c) Give the linearized equations of the system in form of the generic linear system equations
\( \dot{x} = Ax + Bu \).

d) For \( g=9.81 \text{ m/s}^2 \), \( M=0.455 \text{ kg} \), \( m=0.21 \text{ kg} \), \( l=1 \text{ m} \), determine numerically (Matlab “eig()” function) whether the “up” equilibrium point from b) is stable according to a local linear stability analysis. For this purpose, provide a print out of the eigenvalues on which you base your stability analysis, and show the intermediate steps how you derived this result.

e) Assume the control law \( u = -Kx = [-1 \quad -2 \quad 5 \quad 20]x \).

i) Analyze the stability after inserting this control law into \( \dot{x} = Ax + Bu \) and print out the eigenvalues on which you base your analysis.

   From the eigenvalues of the stability analysis, how well do you expect this control law to work, and which features of the behavior can you predict from the eigenvalues? (Short answer in keywords).

ii) Download the cartpole.mdl and CartPoleAnimation.m files. The gains \( K \) are already implemented in this system. Check whether your intuition from the eigenvalues coincides with this simulation (note: the cart-pole animation has a slider window that allows you to change the desired position at which the cart is supposed to be balanced—just play with it). Are you pleased with the performance of this control system? Give reasons for your answer.

f) As you learned in a lecture, there is an optimal way, the linear quadratic regulator (LQR), to computing the gains for the cart-pole system. In order to obtain these optimal gain, execute the following command in Matlab: \( K=lqr(A,B,\text{eye}(4),1) \) and replace the \( K \) in the Simulink simulation with this new gain. The A and B matrices are the matrices that you obtained in c). (Note: you need the Matlab control toolbox for this purpose).

i) Provide a print-out of the new gain matrix \( K \)

ii) Perform the stability analysis from e) for this new control law and print out the eigenvalues of this stability analysis.

iii) How well do you expect the system to work from the eigenvalues, and how well does it work in simulation?